Inductive definitions and proofs

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1 Inductively defined sets

We represent inductively defined sets in the form of a set of inference rules. Suppose we want to define the set S, each inference rule takes the following general form

$$\frac{P_0 \qquad P_1 \qquad \dots \qquad P_n}{a \in S}\ R_0$$

Each premise P_i above the horizontal line is a logical formula that either does not involve S at all or a logical formula of the form $b \in S$ where S does not appear in b.

Each inference rule can be read as an implication that describes the elements that inhabit the set S. Rule R_0, for example, can be read as the proposition $(P_0 \wedge P_1 \dots \wedge P_n) \to a \in S$. When the premise is empty, the statement simply states that the conclusion is true. Note that shuffling the order of $P_0 \dots P_n$ doesn't change the definition of the set S since the underlying proposition should remain equivalent.

Variables that appear free in each rule are implicitly quantified. As an example, recall the following rules we have seen in class.

$$\frac{}{0 \in S}\ R_0$$

$$\frac{n \in S}{n + 4 \in S}\ R_4$$

$$\frac{n \in S}{n + 6 \in S}\ R_6$$

Here, the proposition corresponding to rule R_4 is $\forall n \in S, n + 4 \in S$ where the variable n is universally quantified.

An element a is in the set S if and only if there exists a proof of $a \in S$ using the rules. While it's possible to describe the proofs in English, we can write proofs in the form of *derivation trees*, a formal mathematical object. Here is an example of a derivation tree that shows $10 \in S$.

$$\dfrac{\dfrac{\dfrac{}{0 \in S}\ R_0}{4 \in S}\ R_4}{10 \in S}\ R_6$$

The derivation tree for $10 \in S$ looks more like a sequence than a tree because the rules for the set S only contain at most one premise.

Consider the following inductive definition of the set *beautiful*.

$$\dfrac{}{0 \in \textit{beautiful}}\ B_0$$

$$\dfrac{}{3 \in \textit{beautiful}}\ B_3$$

$$\dfrac{}{5 \in \textit{beautiful}}\ B_5$$

$$\dfrac{n \in \textit{beautiful} \qquad m \in \textit{beautiful}}{m + n \in \textit{beautiful}}\ B_n$$

Here is a derivation tree for $11 \in \textit{beautiful}$.

$$\dfrac{\dfrac{\dfrac{}{3 \in \textit{beautiful}}\ B_3 \quad \dfrac{}{3 \in \textit{beautiful}}\ B_3}{6 \in \textit{beautiful}}\ B_n \quad \dfrac{}{5 \in \textit{beautiful}}\ B_5}{11 \in \textit{beautiful}}\ B_n$$

Since B_n has two premises, invoking B_n to prove $11 \in \textit{beautiful}$ requires us to provide two subproofs/subtrees $6 \in \textit{beautiful}$ and $5 \in \textit{beautiful}$.

Exercise 1. Write a derivation tree for $9 \in \textit{beautiful}$ using only rules B_3 and B_n.

Exercise 2. Write a derivation tree for $9 \in \textit{beautiful}$ that involves at least one usage of rule B_0.

2 Inductive proofs

Given a set S defined by some inference rules, rule induction says that to show that S is a subset of some set R, it suffices to show for each inference rule defining S, the proposition corresponds to the inference rule after replacing S by R holds.

For example, given some set R, to prove that *beautiful* $\in R$, it suffices to show that the propositions correspond to the following rules hold:

$$\frac{}{0 \in R}\ B_0$$

$$\frac{}{3 \in R}\ B_3$$

$$\frac{}{5 \in R}\ B_5$$

$$\frac{n \in R \qquad m \in R}{m + n \in R}\ B_n$$

Thus, to show that $S \subseteq R$, the rule of induction says it is sufficient to prove the following statements.

- $0 \in R$
- $3 \in R$
- $5 \in R$
- $\forall n\, m, n \in R \wedge m \in R \implies m + n \in R$

Exercise 3. Prove by induction that S only contains natural numbers. For each rule, explicitly write down the statement you need to prove and then show why it's true.

Exercise 4. Try proving that S only contains odd numbers, which is a false statement as $3+3 = 6 \in S$. Again, for each rule, write down the statement you need to prove. Which rule fails to hold?

Suppose we want to prove the statement $\forall a \in S, P(a)$ where P is a predicate over objects. We can prove the statement through the induction principle by instantiate R with the set $\{a \mid P(a)\}$. Thus, to prove $\forall a \in S, P(a)$, it suffices to show the following statements.

- $P(0)$ is true.
- $P(3)$ is true.
- $P(5)$ is true.
- $\forall n\, m, P(n) \wedge P(m) \implies P(m + n)$ is true.

For example, to prove that all elements in *beautiful* are linear combinations of 3 and 5, we can instantiate P with $P(a) := \exists m\ n \in \mathbb{N}, a = 3m + 5n$. Then by induction, it suffices to show that the following statements hold.

- 0 is a linear combination of 3 and 5.
- 3 is a linear combination of 3 and 5.
- 5 is a linear combination of 3 and 5.
- If n and m are both linear combinations of 3 and 5, then $m + n$ is a linear combination of 3 and 5.

Don't forget that we are not done yet! The above process helps us find what needs to be proven by invoking the induction principle. We still need to check that all the propositions hold.

As you get more familiar with inductive proofs, you should be able to perform the rewriting from *beautiful* to R in your head and directly prove the statement that corresponds to each rule. With more complicated definitions, however, it is sometimes useful to explicitly write down the induction principle.

Of course, so far we are only talking about how to obtain the induction principle, but we never asked *why* this style of reasoning is correct. Justifying the validity of induction is a topic we will cover later in class.